Analysis of FEM method for coupled 3D-1D diffusion-advection PDEs

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Abstract

This work addresses the numerical analysis of coupled partial differential equations with mixed dimensionality. We formulate a scheme using the finite element method. We prove the existence and uniqueness of the discrete solution, and we also show the convergence of the method by deriving a priori error bound.

Mathematical model

$$-\Delta u + \beta \cdot \nabla u + \kappa|\partial D|(\overline{u} - U)\delta_\Lambda = f, \quad \text{in} \quad \Omega \tag{1a}$$
$$-\frac{d}{ds}\left(|D|\frac{dU}{ds}\right) + \alpha|D|\frac{dU}{ds} - \kappa|\partial D|(\overline{u} - U) = g|D| \quad \text{in} \quad \Lambda \tag{1b}$$
$$u = 0 \quad on \quad \partial\Omega \tag{1c}$$
$$\frac{dU}{ds} = 0 \quad at \quad s = 0 \tag{1d}$$
$$U = 0 \quad at \quad s = L \tag{1e}$$

The unknown parameters are:

- β and α represent velocity, and κ, is the permeability or transfer coefficient.
- $|D(s)|$ is the area cross section and $|\partial D(s)|$ is the perimeter of the cross section at point $s \in (0, L) = \Lambda$.

Motivation

- To formulate a stable numerical scheme that will enable accurate prediction of drug delivery in the blood vessel.

Assumptions

- The parameter $\kappa \in L^\infty(\Lambda)$ is strictly positive and is bounded below by $\kappa_{\min} > 0$.
- There exists positive constants $C_D, C_{\partial D}$ independent of s, such that
$$|D| = C_D(diam(D))^2, \quad |\partial D| = C_{\partial D} diam(D)$$
- Λ, Ω are bounded domains in $\mathbf{R}$ and $\mathbf{R}^3$ respectively and $\Lambda \subset\subset \Omega$.
- $|D(s)| \neq 0$ at any point $s \in (0, L)$, in other words, $\min_s |D(s)| > 0$.

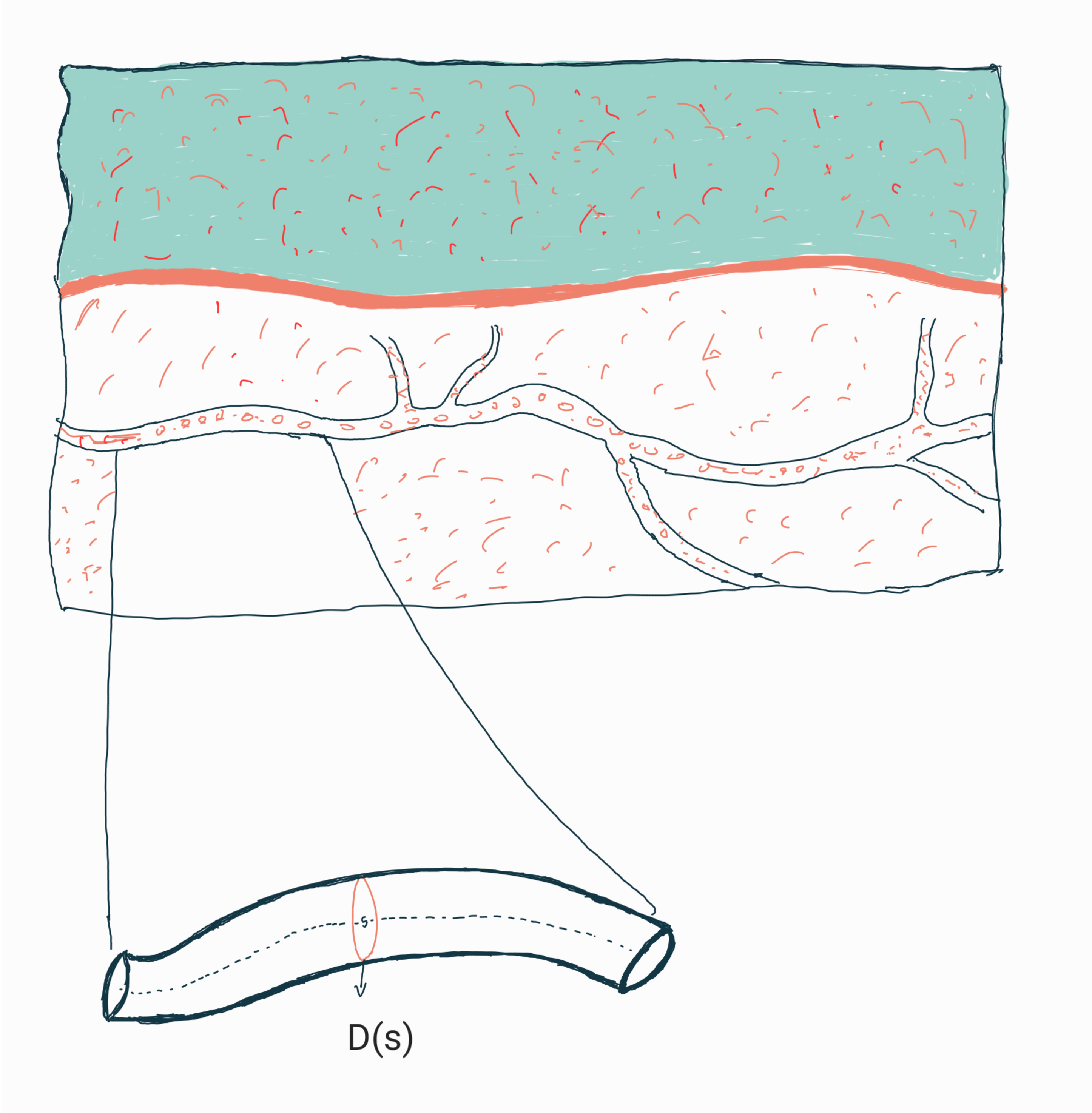
Weak Solution

- The weak form of our PDE is given by
$$\mathcal{A}(\mathcal{U}, \mathcal{V}) = \mathcal{F}(\mathcal{V}),$$
where
$$\mathcal{A}(\mathcal{U}, \mathcal{V}) = a_\Omega(u, v) + a_\Lambda(U, V) + c_\Omega(u, v) + c_\Lambda(U, V) + b_\Lambda(\overline{u} - U, \overline{v} - V) \tag{2}$$
$$\mathcal{F}(\mathcal{V}) = (f, v)_\Omega + (g, V)_{\Lambda,|D|} \tag{3}$$
- We proved the existence and uniqueness of the weak form using the Lax-Milgram Theorem

Bilinear Form

$$a_\Omega(u, v) = (\nabla u, \nabla v)_\Omega \tag{4}$$
$$a_\Lambda(U, V) = (d_s U, d_s V)_{\Lambda,|D|} \tag{5}$$
$$b_\Lambda(u, v) = (\kappa u, v)_{\Lambda,|\partial D|} \tag{6}$$
$$c_\Omega(u, v) = (\beta \cdot \nabla u, v)_\Omega \tag{7}$$
$$c_\Lambda(U, V) = (\alpha d_s U, V)_{\Lambda,|D|} \tag{8}$$

Drug Delivery in the blood Vessel



Numerical Method and Scheme

- Finite Element Method

Let $\mathcal{T}_\Omega^h$ be a partition of domain Ω and $\mathcal{T}_\Lambda^h$ a partition of domain Λ. Define
$$\mathbb{V}_h = V_h^\Omega \times V_h^\Lambda$$
where
$$V_h^\Omega = \{v \in C(\Omega) : v|_E \in \mathbb{P}_{k_1}, \forall E \in \mathcal{T}_\Omega^h \quad \text{and} \quad v = 0 \quad \text{on} \quad \partial\Omega\}$$
and
$$V_h^\Lambda = \{V \in C(\Lambda) : V|_K \in \mathbb{P}_{k_2}, \forall K \in \mathcal{T}_\Lambda^h \quad \text{and} \quad V(L) = 0\}$$
Then, the finite element solution consists of finding
$$\mathcal{U}_h \in \mathbb{V}_h, \quad \mathcal{A}(\mathcal{U}_h, \mathcal{V}_h) = \mathcal{F}(\mathcal{V}_h) \quad \forall \mathcal{V}_h \in \mathbb{V}_h \tag{9}$$

Assumptions

To prove the existence and uniqueness of the finite element solution, we use both the Poincare's and the weighted Poincare's inequality. We also assume the following conditions on β and α :

$$\|\beta\|_{L^\infty(\Omega)} \le \frac{1}{2C_P} \tag{10}$$
$$\|\alpha\|_{L^\infty(\Lambda)} \le \frac{1}{2\hat{C}_{P^*}} \tag{11}$$
$$\|\beta\|_{L^\infty(\Omega)} \le \frac{9}{10(1 + 2C_P)} \tag{12}$$
$$\|\alpha\|_{L^\infty(\Lambda)} \le \frac{9}{10(1 + 2\hat{C}_{p^*})} \tag{13}$$

where C_P and $\hat{C}_{P^*}$ are constants from the Poincare's and weighted Poincare's inequality

Discrete Solution

Lemma 0.1. *Given $f \in L^2(\Omega)$ and $g \in L^2(\Lambda)$. Suppose there exists positive constants C_P and $\hat{C}_P$ such that (10) and (11) are satisfied, then there exists a unique solution to the problem (9).*

Convergence

Theorem 0.2. *Suppose $u \in H^{\frac{3}{2}-\epsilon}(\Omega), U \in H^2(\Lambda)$. Under the assumptions (12) and (13), there exists a constant C independent of u, U and h such that*

$$\|\nabla(u - u_h)\|^2_{L^2(\Omega)} + \||D|^{\frac{1}{2}} d_s(U - U_h)\|^2_{L^2(\Lambda)} + (\||\partial D|^{\frac{1}{2}} \kappa^{\frac{1}{2}}(\overline{u} - \overline{u}_h - (U - U_h))\|^2_{L^2(\Lambda)})$$
$$\le Ch(|u|^2_{H^{\frac{3}{2}-\epsilon}(\Omega)} + |U|^2_{H^2(\Lambda)})$$

Future works

- Formulating a scheme that combines finite element method for the 3D problem and discontinuous Galerkin method for the 1D problem.
- Proving the existence and uniqueness of the discrete solution for the scheme
- Deriving a priori error bound for the combined the method.

Reference

- Federica Laurino and Paolo Zunino. Derivation and analysis of coupled PDEs on manifolds with high dimensionality gap arising from topological model reduction. ESAIM: M2AN, 53(6):2047–2080, 2019.